Assignment 9

Hand in no. 2, 3, 6, and 7 by November 14.

In the following the Initial Value Problem (IVP) refers to $x' = f(t,x),\ x(t_0) = x_0$, where f satisfies the Lipschitz condition in some rectangle containing (t_0, x_0) in its interior, see Notes for details.

1. Solve the IVP for $f(t,x) = \alpha t(1+x^2), \alpha > 0,\ t_0 = 0$, and discuss how the (largest) interval of existence changes as α and x_0 vary.

2. Let x be a solution to the IVP on (c,d), a subinterval of (a,b). Show that it extends to be a solution on $[c,d]$.

3. Let $x_i, i = 1,2$, be two solutions to the same IVP on the subinterval I_i of $[a,b]$. Show that x_1 is equal to x_2 on $I_1 \cap I_2$.

4. Optional. Deduce Picard-Lindelöf Theorem based on the ideas of perturbation of identity. Hint: Take a particular
$$y = \int_{t_0}^{t} f(t, x_0)dt$$
in the relation $x + \Psi(x) = y$.

5. Show that the solution to IVP belongs to C^{k+1} (as long as it exists) provided $f \in C^k$ for $k \geq 1$. In particular, $y \in C^\infty$ provided $f \in C^\infty$.

6. Consider the IVP for second order equation:
$$x'' = f(t, x, x'), \quad x(t_0) = x_0,\ x'(t_0) = x_1\ ,$$
where $f \in C(R), R = [a,b] \times [\alpha, \beta] \times [\gamma, \delta]$. Assume that f satisfies the Lipschitz condition
$$|f(t,x,x') - f(t,y,y')| \leq L(|x-y| + |x'-y'|)\ , \quad (t,x,x'),(t,y,y') \in R\ .$$
Show that the IVP admits a unique solution in $(t_0 - \rho, t_0 + \rho)$ for some $\rho > 0$ by carrying out the following steps.

   (a) Show that the IVP is equivalent to solving
   $$x(t) = x_0 + x_1(t - t_0) + \int_{t_0}^{t} \int_{t_0}^{s} f(r, x(r), x'(r))\, drds\ .$$

   (b) Verify the space $C^1[a,b]$ is complete under the norm
   $$\|x\|_1 = \|x\|_\infty + \|x'\|_\infty\ .$$

   (c) Apply the Contraction Mapping Principle in a closed subset of $(C^1[a,b], \|\cdot\|_1)$.

7. Show that there exists a unique solution h to the integral equation
$$h(x) = 1 + \frac{1}{\pi} \int_{-1}^{1} \frac{1}{1+(x-y)^2} h(y)dy,$$
in $C[-1,1]$. Also show that h is non-negative.

The following passages, which I already covered in class, are extracted from Chapter 4. Here they are enclosed for easy references.

Lemma. Let x be a solution to the IVP above on $[t_0, t_0 + c)$ for some $c \in (t_0, a)$. Suppose that there is $\{t_n\}, t_n \uparrow c$, such that $\lim_{n\to\infty} x(t_n) = x_1$ where (c, x_1) lies in the interior of R. There exists some $\delta > 0$ such that x extends as a solution on $[t_0, c + \delta)$.

Proof.

First, we claim that

$$\lim_{t\uparrow c} x(t) = x_1 \ .$$

For, we have

$$|x(t) - x(t_n)| = \left| \int_{t_n}^{t} f(s, x(s))\, ds \right| \le M|t - t_n| \ .$$

By letting $n \to \infty$, we get $|x(t) - x_1| \le M|t - c|$, from which we deduce $\lim_{t\uparrow c} x(t) = x_1$.

Next, letting $n \to \infty$ in

$$x(t_n) - x(t) = \int_t^{t_n} f(s, x(s))\, ds \ ,$$

we get

$$x(c) - x(t) = \int_t^{c} f(s, x(s))\, ds \ ,$$

which shows that

$$x'(c) = \lim_{t\uparrow c} \frac{f(c) - x(t)}{c - t} = f(t, x(c)).$$

Hence x is differentiable at c (more precisely, left derivative exists) and satisfies the differential equation.

Finally, since (c, x_1) sits in the interior of R, we may apply Picard-Lindelöf Theorem to a small rectangle inside R centered at (c, x_1) to get a solution y to the same differential equation on $(c-\delta, c+\delta)$ for small δ. It is clear the function $z(t) = x(t), t \in [t_0, c)$, and $z(t) = y(t)\ ,\ t \in [c, c+\delta)$ defines a solution of the IVP extending x.

Proposition. Under the setting of Picard-Lindelöf Theorem, the unique solution exists on the interval $[t_0 - a^*, t_0 + a^*]$ where

$$a^* = \min\left\{a, \frac{b}{M}\right\} \ .$$

Proof. We will prove the solution exists on $[t_0, t_0 + a^*)$. Similarly one can show that it exists on $(t_0 - a^*, t_0]$. Let

$$c^* = \sup\{c : \text{ there exists a solution on } [t_0, t_0 + c] \ .\}$$

Then the solution is well-defined on $[t_0, t_0 + c^*)$. If $c^* = a$, then the solution exists on $[t_0, t_0 + a)$ and hence on $[t_0, a^*)$. Let us assume $c^* < a$. In view of lemma above, there is no sequence $t_n \uparrow c^*$ such that $(t_n, x(t_n))$ converges to an interior point of R. Since $c^* < a$, $x(t)$ must either converge to $x_0 + b$ or $x_0 - b$. Let us assume it is the former. The proof is the same when the latter holds. Letting $n \to \infty$ in the relation

$$x(t_n) - x_0 = \int_{t_)}^{t_n} f(s, x(s))\, ds \ ,$$

we obtain

$$b = \left| \int_{t_0}^{t_0+c^*} f(s, x(s))\, ds \right| \le Mc^* \ ,$$

which implies $c^* \ge b/M$. Hence the solution x exists on $[t_0, b/M)$.

According to Problem 2, the solution in fact exists on $[t_0 - a^*, t_0 + a^*]$.